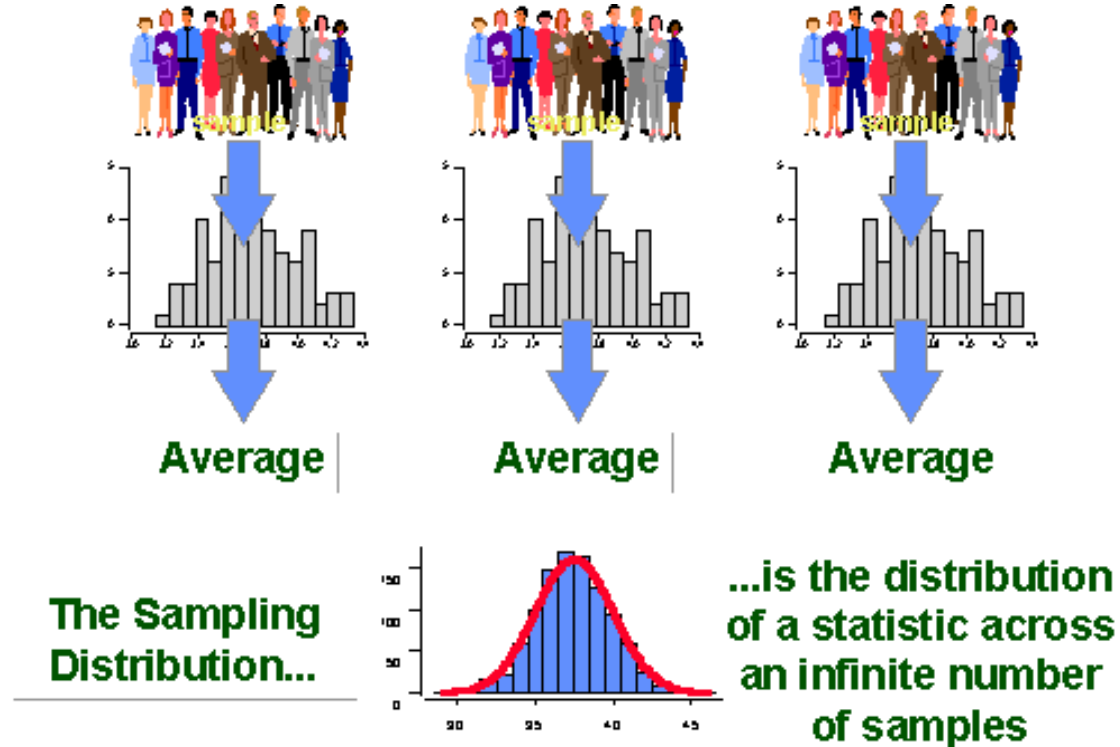


# Chapter 4: Probability and Counting Rules

*Before we can move from descriptive statistics to inferential statistics, we need to have some understanding of probability:*



## Section 4-1: Sample Spaces and Probability

**Probability** - the likelihood of an event occurring.

**Probability experiment** – a chance process that leads to well-defined results called outcomes. (i.e., some mechanism that produces a set of outcomes in a random way).

**Outcome** – the result of a single trial of a probability experiment.

**Example: Roll a die once. What could happen in one roll of the die?**

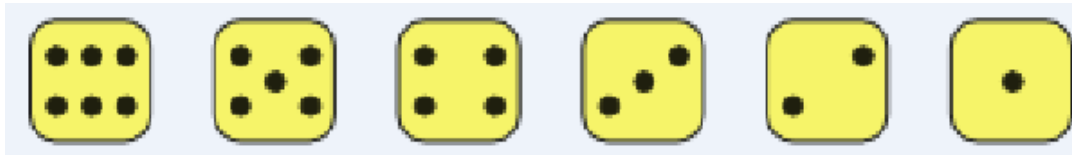
**Sample space** – the set of all possible outcomes of a probability experiment.

**Example: What is the sample space for one flip of a coin?**



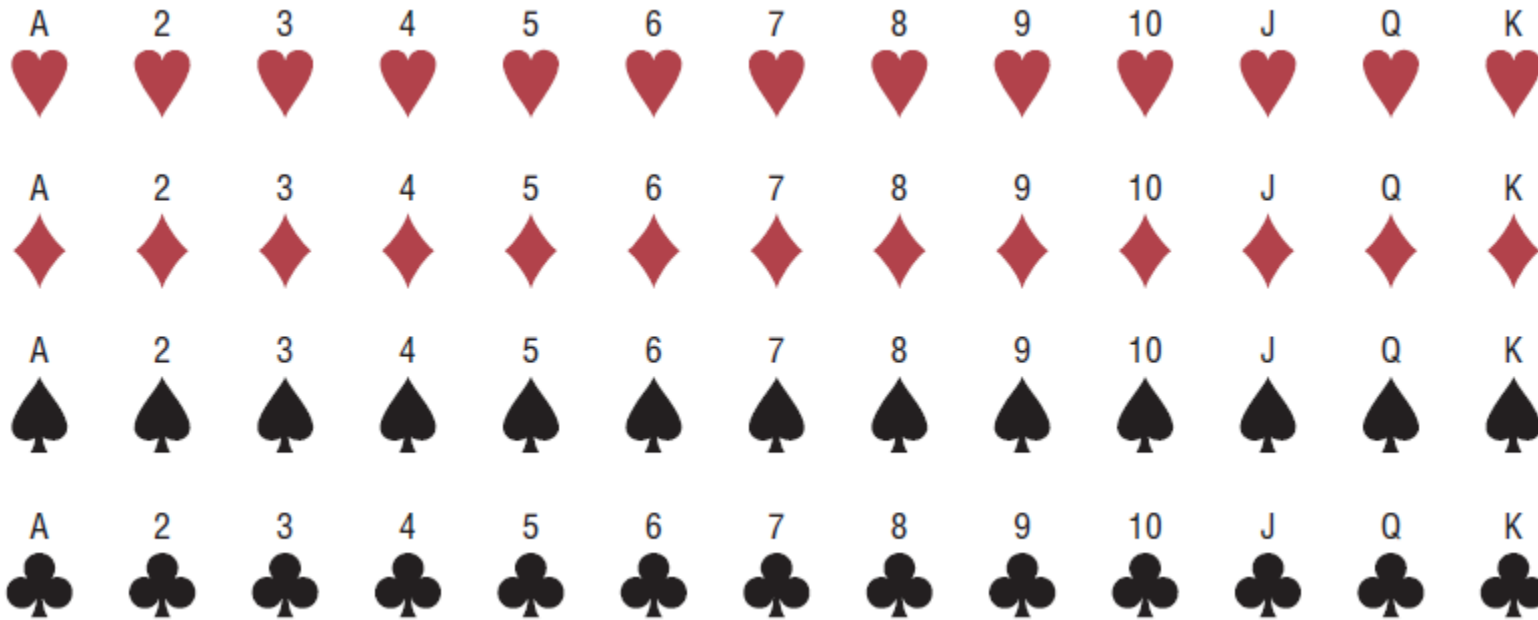
**Heads, Tails**

**Example: Suppose I roll two six-sided dice. What is the sample space for the possible outcomes?**



**1, 2, 3, 4, 5, 6**

**Example:** Find the sample space for drawing one card from an ordinary deck of cards.



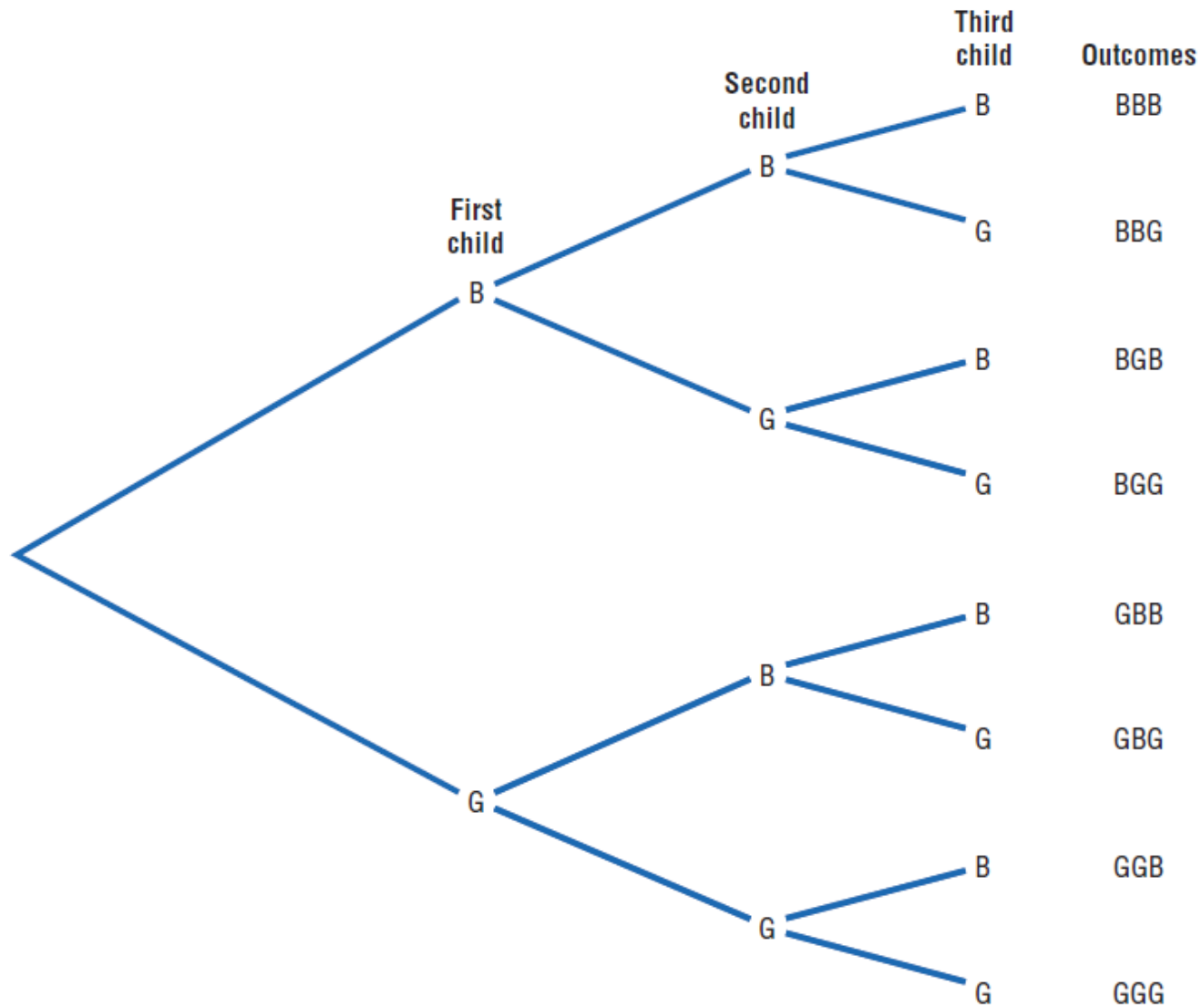
**Sample space consists of all possible  $13 \times 4 = 52$  outcomes:**  
 **$A♥, 2♥, \dots, K♥, \dots, A♣, 2♣, \dots, K♣$**

**TREE DIAGRAM – a device consisting of line segments emanating from a starting point and also from the outcome points. It is used to determine all possible outcomes of a probability experiment.**

Example: Use a tree diagram to find the sample space for the sex of three children in a family.

Our outcome pertains to the sex of one child AND the second of the next child AND the sex of the third child. Each of the children will correspond to a branching in the tree.

- What is the sex of the first child? Boy/Girl
- Given the sex of the first child, what is the sex of the second child?
- Given the sex of the first two children, what is the sex of the third child?



Example: 3 pairs of jeans, 5 shirts, 2 hats. Use a tree diagram to determine all possible outfits composed of a pair of jeans, shirt, and a hat.

**Event** – consists of a set of possible outcomes of a probability experiment.

- Can be one outcome or more than one outcome.
- **Simple event** – an event with one outcome.
- **Compound event** – an event with more than one outcome.

Example: Roll a die and get a 6 (**simple event**).

Example: Roll a die and get an even number (**compound event**).



There are three basic interpretations of probability:

1. Classical probability
2. Experimental or relative frequency probability
3. Subjective probability

**Theoretical (Classical) Probability** – uses sample spaces to determine the numerical probability that an event will happen.

- We do not actually perform the experiment to determine the theoretical probability.
- Assumes that all outcomes are equally likely to occur.

## Formula for Classic Probability

The probability of an event  $E$  is

$$P(E) = \frac{\text{Number of outcomes in } E}{\text{Number of outcomes in the sample space}} = \frac{n(E)}{n(S)}$$

where  $S$  denotes the sample space and  $n(\cdot)$  means “the number of outcomes in . . . .”

**Rounding Rules for Probabilities** – probabilities should be expressed as reduced fractions or rounded to 2-3 decimal places. If the probability is extremely small then round to the first nonzero digit.

Example: Consider a standard deck of 52 cards:

Find the probability of selecting a queen

$$P(\text{queen}) = \frac{4}{52} = \frac{1}{13} = 0.077$$

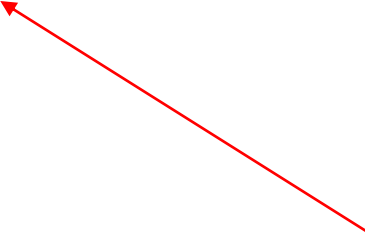
CAN LEAVE AS A  
REDUCED FRACTION!  
This is to demonstrate  
rounding.

Find the probability of selecting a spade,  $P(\text{spade}) =$

Find the probability of selecting a red ace,  $P(\text{red ace}) =$

## Probability Rules

1. The Probability of an event  $E$  must be a number between 0 and 1. i.e.,  $0 \leq P(E) \leq 1$ .
2. If an event  $E$  **cannot** occur, then its probability is **0**.
3. If an event  $E$  **must** occur, then its probability is **1**.
4. The sum of all probabilities of all the outcomes in the sample space is 1.



Always a good  
“sanity check” when  
doing calculations!

Example: Suppose I roll a standard six-sided die.

What's the probability I get a 7?

What's the probability that I get a number less than 7?

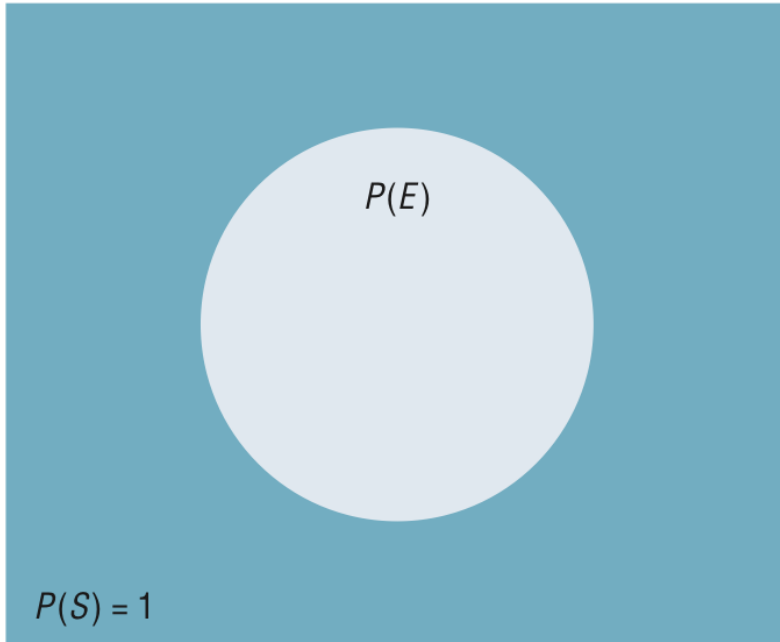
What's the probability that I get a 1 or a 2 or a 3 or a 4 or a 5 or a 6?

## Complementary Events

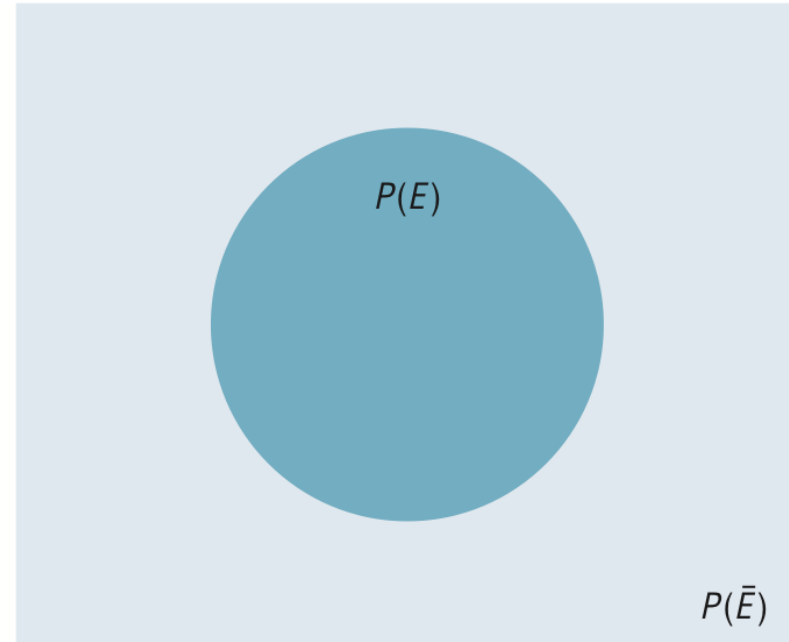
**Complement of an event  $E$**  - the set of outcomes in the sample space that are **not** included in the outcomes of event  $E$ . The complement of  $E$  is denoted by  $\bar{E}$  (“E bar”).

Note: The outcomes of an event and the outcomes of the complement make up the entire sample space.

**Venn Diagram** – a visual way of representing probabilities.



(a) Simple probability



(b)  $P(\bar{E}) = 1 - P(E)$

Venn diagrams are a wonderful tool to help think through probability calculations.

Example: What is the complement of the following events?

Rolling a six-sided die and getting a 4?

*Complement = Rolling a die and getting 1, 2, 3, 5 or 6.*

Rolling a die and getting a multiple of 3?

Selecting a day of the week and getting a weekday?

Selecting a month and getting a month that begins with an A?



## Rule for Complementary Events:

$$P(\bar{E}) = 1 - P(E) \text{ or } P(E) = 1 - P(\bar{E}) \text{ or } P(E) + P(\bar{E}) = 1.$$

Example: The probability of purchasing a defective light bulb is 12%. What is the probability of not purchasing a defective light bulb?

$$P(\textit{not defective}) = 1 - P(\textit{defective}) = 1 - 0.12 = 0.88$$

Example: What is the probability of not selecting a club in a standard deck of 52 cards?

**Empirical Probability** – the relative frequency of an event occurring from a probability experiment over the long-run.

- It relies on **actual experience** to determine the likelihood of an outcome rather than assuming equally likely outcomes.

Given a frequency distribution, the probability of an event being in a given class is:

$$P(E) = \frac{\text{frequency for the class}}{\text{total frequencies in the distribution}} = \frac{f}{n}.$$

This probability is called the **empirical probability**.

Example: Observe the proportion of male babies out of many, many births.

Example: Major Field of Study

<u>Class</u>	<u>Frequency</u>
Math	5
History	7
English	4
Science	<u>9</u>
	25

What is the probability of being a math major? Science major? History or English Major?

$$P(M) = 5/25=0.2$$

$$P(S) =$$

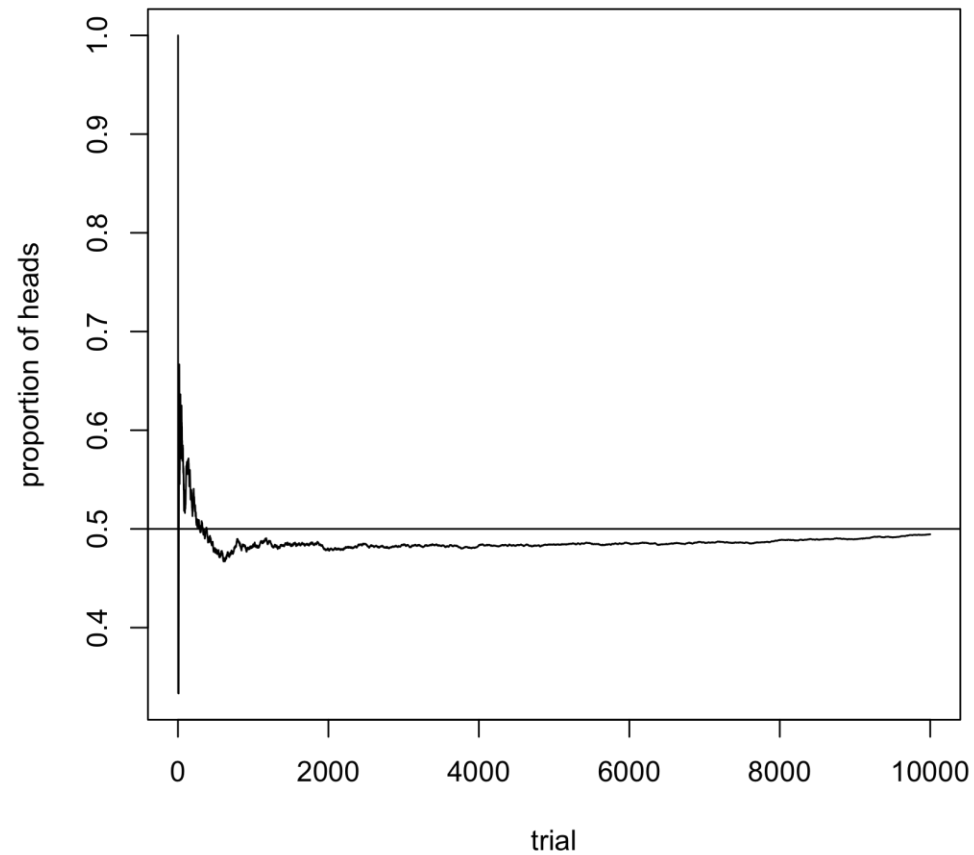
$$P(H \text{ or } E) =$$

The **Law of Large Numbers** tells us that as the number of trials increases the empirical probability gets closer to the theoretical (true) probability.

Because of the law of large numbers we will interpret the probability to be the “long-run” results (which we know approximates the theoretical probability).

The **probability** of a particular outcome is the proportion of times the outcome would occur in a long-run of observations.

## Example: Proportion of times a fair coin comes up as a “head”



**Subjective Probability** – uses a probability value based on an educated guess or estimate, employing opinions and inexact information.

- Often, you cannot “repeat” the probability experiment.

Example: What is the probability you will pass this class?

Example: What is the probability that you will get a certain job when you apply?

*Read about Probability and Risk Taking on pg. 194-195. We are notoriously bad at subjectively estimating probability!*

## SECTION 4-2: THE ADDITION RULES FOR PROBABILITY

There are times when we want to find the probability of two or more events. For example, when selecting a card from a deck we may want to find the probability of selecting a card that is a four or red. In this case there are 3 possibilities to consider:

- The card is a four
- The card is red
- The card is a four and red

Now consider selecting a card and we want to find the probability of selecting a card that is a spade or a diamond. In this case there are only 2 possibilities to consider:

- The card is a spade
- The card is a diamond

Notice it can't be both a spade and a diamond.

**Mutually exclusive** - Two events are mutually exclusive (disjoint) if they cannot occur at the same time.

*Looking ahead: If we have mutually exclusive events, then their probabilities will add. Let's make sure we understand what it means for events to be mutually exclusive.*

**Example:**

Which events are mutually exclusive and which are not, when a single die is rolled?

1. Getting an odd number and getting an even number  
*Mutually exclusive! You can't have a roll be both.*

2. Getting a 3 and getting an odd number

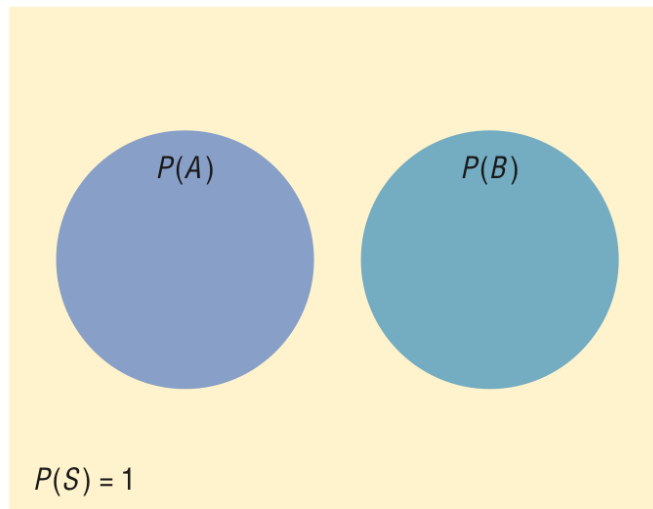
3. Getting an odd number and getting a number less than 4

4. Getting a number greater than 4 and getting a number less than 4

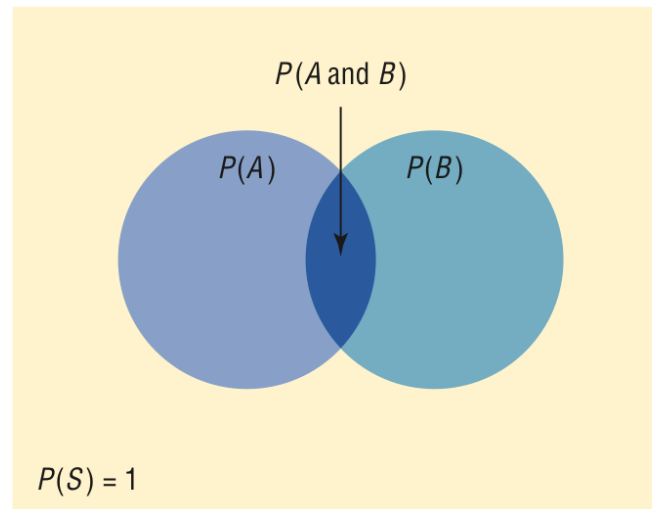


**Intersection** – the intersection of events  $A$  and  $B$  are the outcomes that are in both  $A$  and  $B$ . If  $A$  and  $B$  have outcomes intersecting each other then we say that they are **non-mutually exclusive**.

**Union** – the union of events  $A$  and  $B$  are all the outcomes that are in  $A$ ,  $B$ , or both.

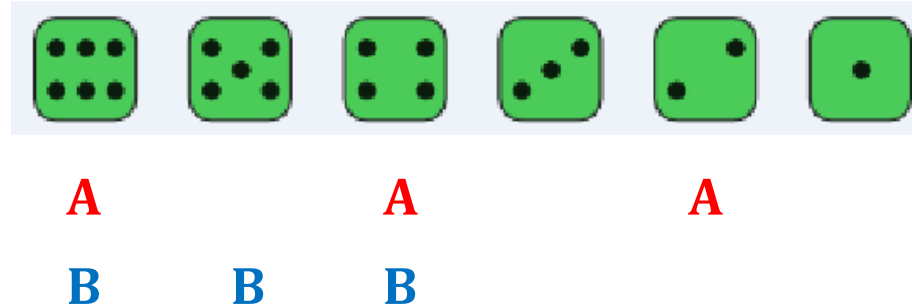


(a) Mutually exclusive events  
 $P(A \text{ or } B) = P(A) + P(B)$



(b) Nonmutually exclusive events  
 $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$

Example: Suppose we roll a six-sided die. Let  $A$  be that we roll an even number. Let  $B$  be that we roll a number greater than 3.



What is the intersection between  $A$  and  $B$ ?

*Rolling a 6 or 4*

What is the union of  $A$  and  $B$ ?

*Rolling a 6, 5, 4, or 2*

## Addition Rules (These apply to “or” statements.)

**Rule 1:** If two events  $A$  and  $B$  are mutually exclusive, then:

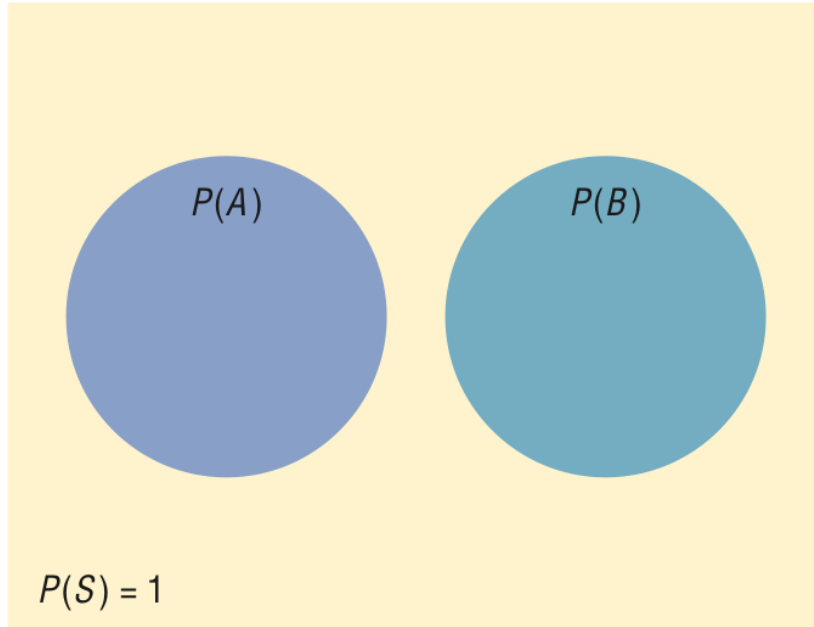
$$P(A \text{ or } B) = P(A) + P(B)$$

**Rule 2:** For **ANY** two outcomes  $A$  and  $B$ ,

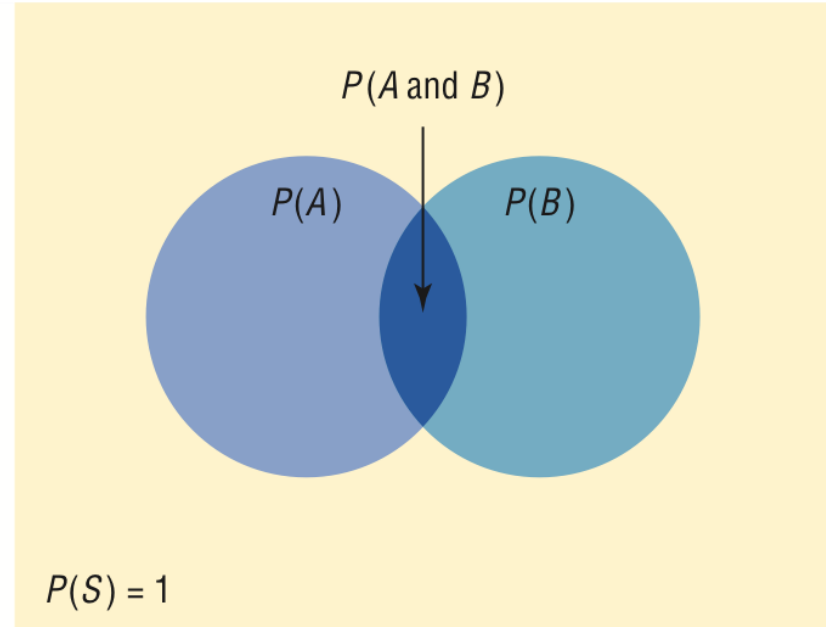
$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

*Note:* In probability “ $A$  or  $B$ ” denotes that  $A$  occurs, or  $B$  occurs, or both occur!

# Venn diagrams for mutually versus non-mutually exclusive events:



**(a)** Mutually exclusive events  
 $P(A \text{ or } B) = P(A) + P(B)$



**(b)** Nonmutually exclusive events  
 $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$

**Example:** At a political rally, there are 20 Republicans, 13 Democrats, and 6 Independents. If a person is selected at random, find the probability that he or she is either a Democrat or an Independent.

Event A = a person is a democrat

Event B = a person is an independent

These are mutually exclusive since you can NOT be both.

$$\begin{aligned} P(\text{a person is a Democrat or an Independent}) &= P(A \text{ or } B) \\ &= P(A) + P(B) \\ &= \frac{13}{20+13+6} + \frac{6}{20+13+6} \\ &= \frac{13}{39} + \frac{6}{39} \\ &= \frac{19}{39} \approx 0.487 \end{aligned}$$

**Example:** A single card is drawn at random from an ordinary deck of cards. Find the probability that the card is either an ace or a red card. (*Hint: Define events. Determine if mutually exclusive. Use appropriate rule on slide 28.*)

**Example:** On New Year's Eve, the probability of a person driving while intoxicated is 0.32, the probability of a person having a driving accident is 0.09, and the probability of a person having a driving accident while intoxicated is 0.06. What is the probability of a person driving while intoxicated or having a driving accident?

## Section 4-3 The Multiplication Rules and Conditional Probability

The multiplication rules can be used to find the probabilities of two or more events that occur in sequence.

- When there are multiple stages in the experiment
- When there are 2 or more trials
- For example, tossing a coin then rolling a die (a 2-stage experiment)

*Multiplication rules apply to “**and**” statements.*



**Independent** - two events  $A$  and  $B$  are independent events if the fact that  $A$  occurs does not affect the probability of  $B$  occurring.

Example: Rolling one die and getting a six, rolling a second die and getting a three.

Example: Draw a card from a deck and replacing it, drawing a second card from the deck and getting a queen.

*In each example, the first event has no effect on the probability of the second event.*

## Multiplication Rule for Independent Events

**Multiplication Rule 1:** When two events  $A$  and  $B$  are independent, then  $P(A \text{ and } B) = P(A)P(B)$

*That is, when events are independent, their probabilities multiply in an “and” statement.*

Example: The New York state lottery uses balls numbered 0-9 circulating in 3 separation bins. To select the winning sequence, one ball is chosen at random from each bin. What is the probability that the sequence 9-1-1 would be the one selected?

$$P(\text{Sequence } 9-1-1) = \frac{1}{10} \times \frac{1}{10} \times \frac{1}{10} = \frac{1}{1000} = 0.001$$

*Actually, this is the same probability of any of the equally likely 1000 draws*

Example: Approximately 9% of men have a type of color blindness that prevents them from distinguishing between red and green. If 3 men are selected at random, find the probability that all of them will have this type of red-green color blindness.

**Dependent** - Two outcomes are said to be *dependent* if knowing that one of the outcomes has occurred affects the probability that the other occurs.

Examples:

- Drawing a card from a deck, not replacing it, and then drawing a second card.
- Being a lifeguard and getting a suntan
- Having high grades and getting a scholarship
- Parking in a no-parking zone and getting a ticket

**The Conditional Probability** of an event  $B$  in relationship to an event  $A$  is the probability that event  $B$  occurs after event  $A$  has already occurred.

- This probability is denoted as  $P(B | A)$ .

When events are dependent we cannot use the current form of the multiplication rule. We have to modify it.

## Multiplication Rule for Dependent Events

**Multiplication Rule 2:** When two events are dependent, the probability of both occurring is  $P(A \text{ and } B) = P(A)P(B | A)$ .

Example: What is the probability of getting an Ace on the first draw and a king on a second draw?

$$P(\text{Ace then King}) = P(\text{Ace})P(\text{King} | \text{Ace})$$

$$= \frac{4}{52} \times \frac{4}{51} \approx 0.006$$

First draw from full deck of 52 cards has 4 Aces

Second draw from a deck of 51 cards (which is missing a single Ace) has 4 Kings

Example: At a university in western Pennsylvania, there were 5 burglaries reported in 2003, 16 in 2004, and 32 in 2005. If a researcher wishes to select two burglaries at random to further investigate, find the probability that both will have occurred in 2004?

Example: World Wide Insurance Company found that 53% of the residents of a city had homeowner's insurance (H) with the company. Of these clients, 27% also had automobile insurance (A) with the company. If a resident is selected at random, find the probability that the resident has both homeowner's insurance and automobile insurance with World Wide Insurance Company.

$$\begin{aligned}P(H \text{ and } A) &= P(H)P(A | H) \\ &= 0.53 \times 0.27 \\ &= 0.1431\end{aligned}$$



## Formula for Conditional Probability

The probability that the second event  $B$  occurs given that the first event  $A$  has occurred can be found by dividing the probability that both events have occurred by the probability that the first event has occurred. For events  $A$  and  $B$ , the conditional probability of event  $B$  given  $A$  occurred is

$$P(B | A) = \frac{P(A \text{ and } B)}{P(A)}$$

Example: A box contains black chips and white chips. A person selects two chips without replacement. If the probability of selecting a black chip **and** a white chip is  $15/56$ , and the probability of selecting a black chip in the first draw is  $3/8$ , find the probability of selecting the white chip on the second draw, **given** that the first chip selected was a black chip.

**Want to compute:**  $P(\text{White chip on second draw} | \text{First chip was black})$

**Know:**  $P(\text{Selecting black and white chip}) = 15/56$

$P(\text{Selecting black chip on first draw}) = 3/8$

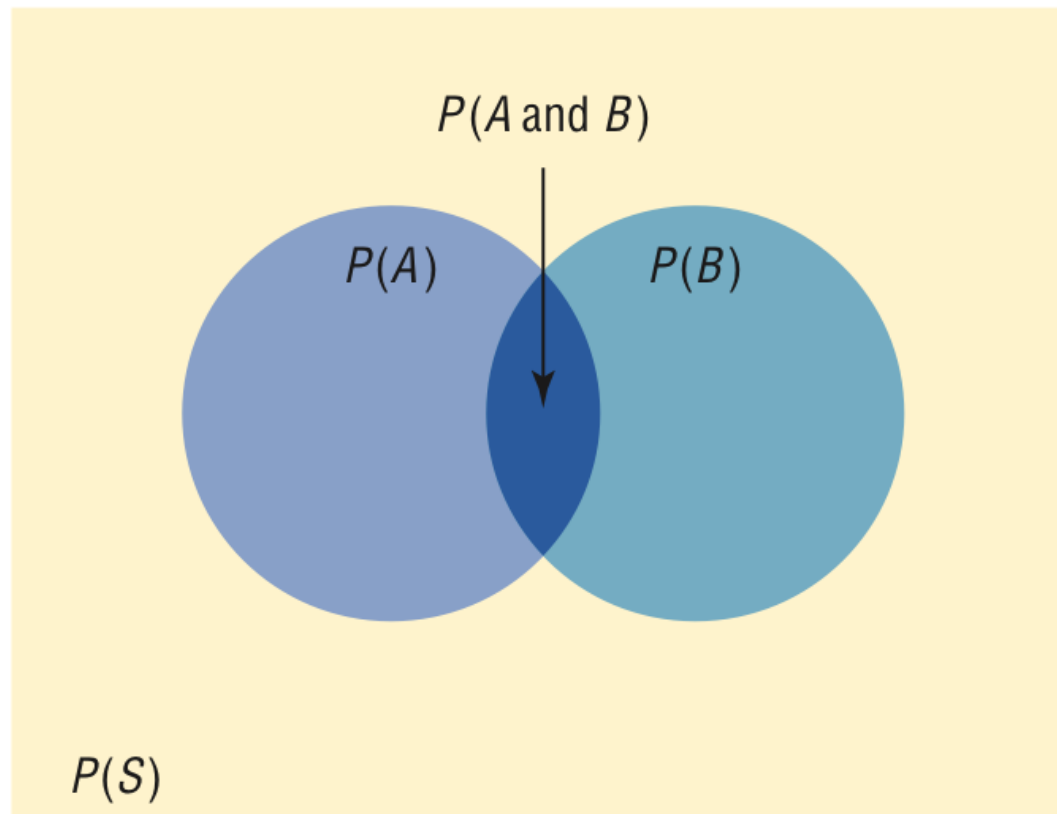
**Applying formula for conditional probability:**

$$\begin{aligned} &P(\text{White chip on second draw} | \text{First chip was black}) \\ &= \frac{P(\text{Selecting black and white chip})}{P(\text{First chip was black})} = \frac{15/56}{3/8} \approx 0.714 \end{aligned}$$

Example: The probability that Sam parks in a no-parking zone *and* gets a parking ticket is 0.09. The probability that Sam cannot find a legal parking space and has to park in the no-parking zone is 0.20. Today, Sam arrived at UCD and has to park in a no-parking zone. Find the probability that he will get a parking ticket.

Let  $N$  = parking in a no-parking zone,  $T$  = getting a ticket

## Venn Diagram for Conditional Probability



$$P(B|A) = \frac{P(A \text{ and } B)}{P(A)}$$

## Probabilities for “At Least”

The multiplication rules can be coupled with the complimentary event rules (Section 4-1) to solve probability problems involving “at least”.

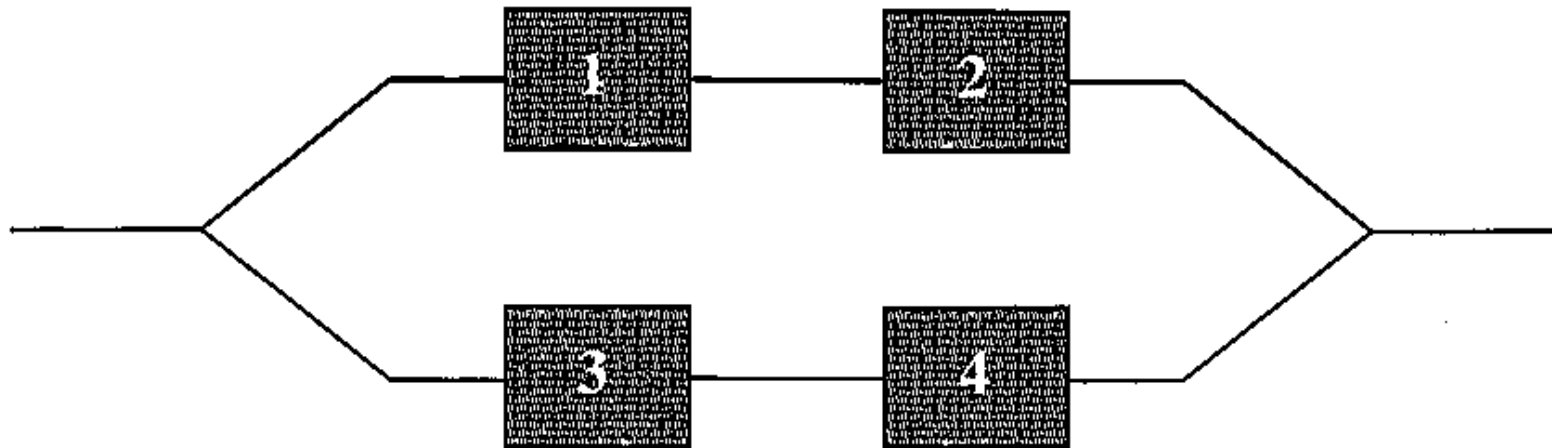
Example: A game is played by drawing 4 cards from an ordinary deck and replacing each card after it is drawn. Find the probability that at least 1 ace is drawn.

$$\begin{aligned} P(\text{at least 1 Ace}) &= 1 - P(\text{no aces drawn}) && \longleftarrow \text{Complementation} \\ &= 1 - \frac{48}{52} \times \frac{48}{52} \times \frac{48}{52} \times \frac{48}{52} && \longleftarrow \text{Multiplication Rule} \\ &= 1 - 0.726025 && \text{Note rounding to 3 decimal} \\ &\approx 0.274 && \longleftarrow \text{places.} \end{aligned}$$

What keyword in a probability problem probably means you should use the additional rules?

What keyword in a probability problem probably means you should use the multiplication rules?

Example: Consider a system of four components, as pictured in the diagram. Components 1 and 2 form a series subsystem, as do Components 3 and 4. The two subsystems are connected in parallel. Suppose that  $P(1 \text{ works})=.9$ ,  $P(2 \text{ works})=.9$ ,  $P(3 \text{ works})=.9$ ,  $P(4 \text{ works})=.9$ , and that the four outcomes are independent of each other.



The subsystem works only if **both** components work. What is the probability the 1-2 subsystem works?

What is the probability that the 1-2 subsystem doesn't work?  
That the 3-4 subsystem doesn't work?

The system won't work if both the 1-2 subsystem doesn't work and the 3-4 subsystem doesn't work. What is the probability that it won't work? That it will work?



## SUMMARY OF PROBABILITY RULES

**Addition Rule 1:** If two events  $A$  and  $B$  are mutually exclusive, then  $P(A \text{ or } B) = P(A) + P(B)$

**Addition Rule 2:** For **ANY** two outcomes  $A$  and  $B$ ,  
 $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$

**Multiplication Rule 1:** When two events  $A$  and  $B$  are independent, then  $P(A \text{ and } B) = P(A)P(B)$

**Multiplication Rule 2:** When two events are dependent, the probability of both occurring is  $P(A \text{ and } B) = P(A)P(B | A)$ .

$Pr(\text{at least } 1) = 1 - Pr(\text{none})$   
 $P(A \text{ and } B) = P(A)P(B | A)$ .

$$P(B | A) = \frac{P(A \text{ and } B)}{P(A)}$$